CSCI 400: Solutions to λ-Calculus Examples

Jonathan Sumner Evans February 14, 2019

1 Alpha Equivalence

Identify if each of the following are a valid α-conversion.

1. $\lambda x.\lambda x.x \to \lambda y.\lambda y.y$

   This is a valid α-conversion. We rename all of the instances of x to y.

2. $\underbrace{\lambda x}_{1}.\underbrace{\lambda x}_{2}.\underbrace{x}_{3} \to \lambda y.\lambda x.x$

   This is a valid alpha conversion. This is because the x (3) is bound by the second abstraction (2), **not** by the first abstraction (1). Thus, by renaming the first x to y, we end up with the term on the right.

   The act of rebinding a variable is called "variable shadowing". You might want to know that for the test.

3. $\lambda x.\underbrace{\lambda x}_{2}.\underbrace{x}_{3} \to \underbrace{\lambda y}_{1}.\lambda x.y$

   This is **not** a valid α-conversion. The reason for this is similar to the second problem. The x (3) is bound to the second abstraction (2) on the left hand side, however it is bound to the outer abstraction (1) on the right hand side.

4. $\lambda x.\lambda y.x \to \lambda y.\lambda y.y$

   This is **not** a valid α-conversion. The reason is that this introduces a **naming conflict**.

2 Beta Reductions

Fully β-reduce each of the following expressions:

5. In this example, I am showing the full currying steps for each application. Each of the applications only accepts a single term.

$$
\begin{aligned}
&(\lambda x.\lambda y.\lambda f.fxy)\overbrace{(\lambda x.\lambda y.y)}^{x}(\lambda x.\lambda y.x)(\lambda x.\lambda y.y)\\
\rightsquigarrow&(\lambda y.\lambda f.f(\lambda x.\lambda y.y)y)\overbrace{(\lambda x.\lambda y.x)}^{y}(\lambda x.\lambda y.y)\\
\rightsquigarrow&(\lambda f.f(\lambda x.\lambda y.y)(\lambda x.\lambda y.x))\overbrace{(\lambda x.\lambda y.y)}^{f}\\
\rightsquigarrow&(\lambda x.\lambda y.y)\overbrace{(\lambda x.\lambda y.y)}^{x}(\lambda x.\lambda y.x)\\
\rightsquigarrow&(\lambda y.y)\overbrace{(\lambda x.\lambda y.x)}^{y}\\
\rightsquigarrow&(\lambda x.\lambda y.x).
\end{aligned}
$$

6. In this example, I am not showing the full currying steps for each application.

$$
\begin{aligned}
&(\lambda a.\lambda b.a(\lambda b.\lambda f.\lambda x.f(bfx))b)\overbrace{(\lambda f.\lambda x.fx)}^{a}\overbrace{(\lambda f.\lambda x.f(fx))}^{b}\\
\rightsquigarrow&(\lambda f.\lambda x.fx)\overbrace{(\lambda b.\lambda f.\lambda x.f(bfx))}^{f}\overbrace{(\lambda f.\lambda x.f(fx))}^{x}\\
\rightsquigarrow&(\lambda b.\lambda f.\lambda x.f(bfx))\overbrace{(\lambda f.\lambda x.f(fx))}^{b}\\
\rightsquigarrow&(\lambda f.\lambda x.f((\lambda f.\lambda x.f(fx))fx))\\
\rightsquigarrow&(\lambda f.\lambda x.f((\lambda f.\lambda x.f(fx))\overbrace{f}^{f}\overbrace{x}^{x}))\\
\rightsquigarrow&(\lambda f.\lambda x.f(f(fx))).
\end{aligned}
$$