Symbolic Computation

Principles of Programming Languages

Colorado School of Mines

https://lambda.mines.edu
1. What questions did you have on the reading? Can your group members answer, or you can ask me.
2. Define symbolic computation in your own words.
3. What structures in Racket would you find useful for symbolic computation?
4. Share what other applications you came up with for symbolic computation. Formulate some more with your group.
Wikipedia considers symbolic computation to be simply *computer algebra*.

While computer algebra is a form of symbolic computation, there are plenty of other applications.

- Programming languages
- Compilers
- Artificial intelligence
- ...
Lisp dialects have a **homoiconic syntax**: the code is data, and data is code. Lists being the structure of the language syntax, code can be manipulated just like lists.

- The concept of "quoting" is fairly unique to just Lisp.
- It leads to a natural way to manipulate and work on code in the language.
- **Key point:** we can manipulate code before it is evaluated!

**John McCarthy (1958)**

*Recursive Functions of Symbolic Expressions and their Computation by Machine*

- Today we will explore a practical application of symbolic computation in artificial intelligence.
Boolean Expressions as S-Expressions

To represent boolean expressions in Racket, we need to formalize an s-expression syntax for them:

<table>
<thead>
<tr>
<th>Conjunction</th>
<th>$a \land b \land c \ldots$</th>
<th>$(\text{and } a \ b \ c \ldots)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjunction</td>
<td>$a \lor b \lor c \ldots$</td>
<td>$(\text{or } a \ b \ c \ldots)$</td>
</tr>
<tr>
<td>Negation</td>
<td>$\neg a$</td>
<td>$(\text{not } a)$</td>
</tr>
</tbody>
</table>

Practice: convert to s-expression

1. $a \land (b \lor c \lor d) \land d$
2. $\neg a \land (a \lor \neg b) \land \neg (a \lor b)$
Note

Depending on your background, you may already know this. Bear with me while I explain it to everyone else.

A boolean expression is in **conjunctive normal form** (CNF) if and only if all of the below are true:

- It only contains conjunctions, disjunctions, and negations.
- Negations only contain a variable, not a conjunction or disjunction.
- Disjunctions only contain variables and negations.

Example:

\[(a \lor b \lor c) \land (\neg a \lor b)\]
(define/match (in-cnf? expr [level 'root])
  ((((? symbol?) _) #t)
  [(
    (not ,((? symbol?) _)) _ #t)
  [((list-rest 'or args) (or 'root 'and))
    (andmap (λ (x) (in-cnf? x 'or)) args)]
  [((list-rest 'and args) 'root)
    (andmap (λ (x) (in-cnf? x 'and)) args)]
  [(- -) #f])
)
We can convert any boolean expression composed of just conjunctions, disjunctions, and negations to CNF using the following mathematical properties:

- **Elimination of double-negation:** \( \neg\neg a \rightarrow a \)
- **DeMorgan’s Law (Conjunction):** \( \neg(a \land b) \rightarrow (\neg a \lor \neg b) \)
- **DeMorgan’s Law (Disjunction):** \( \neg(a \lor b) \rightarrow (\neg a \land \neg b) \)
- **Distributive Property:** \( a \lor (b \land c) \rightarrow (a \lor b) \land (a \lor c) \)
Practice: Convert to CNF

Convert each expression to CNF:

1. \( \neg(a \land \neg b) \)
2. \( \neg((a \lor b) \land \neg(c \lor d)) \)
3. \( \neg((a \lor b) \land (c \lor d)) \)
Here’s the base structure we want our code to follow:

```
(define (boolean->cnf expr)
 (if (in-cnf? expr)
     expr
     (boolean->cnf
      (match expr
       (...))))) ;; cases for the conversions we know
```
Double Negation Pattern Match

`\((\neg (\neg e)) e\)\]
Simplify and/or of single argument

\[
\text{\texttt{[`\texttt{(or \ ,e) e}]}}
\]
\[
\text{\texttt{[`\texttt{(and \ ,e) e}]}}
\]
DeMorgan’s Law

- DeMorgan’s Law for Conjunction
  
  ```
  `(not (and ,@ (list-rest args)))
  `(or ,@ (map (curry list 'not) args))
  ```

- DeMorgan’s Law for Disjunction
  
  ```
  `(not (or ,@ (list-rest args)))
  `(and ,@ (map (curry list 'not) args))
  ```
Explosion of and/or with nested expression

- and in and simplification
  \[
  `(\text{and} ,\text{@}(\text{list-no-order} (\text{list-rest} \ '\text{and} \ \text{inside}) \ \text{outside} \ ...)) \\
  `(\text{and} ,\text{@}\text{inside} ,\text{@}\text{outside})]
  \]

- or in or simplification
  \[
  `(\text{or} ,\text{@}(\text{list-no-order} (\text{list-rest} \ '\text{or} \ \text{inside}) \ \text{outside} \ ...)) \\
  `(\text{or} ,\text{@}\text{inside} ,\text{@}\text{outside})]
  \]
Distributive Law

\[
\text{\texttt{`(or ,[]} (\texttt{list-no-order (list-rest \textquotesingle{}and and-args) args ...}))}
\text{ \texttt{`(or ,[]} (\texttt{cdr args})}
\text{ \texttt{(and ,[]} (\texttt{map}
\text{ \texttt{(\lambda (x) (list \textquotesingle{}or (car args) x))}
\text{ \texttt{and-args))}))]]
\]
Recurse otherwise...

\[
\text{[(list-rest sym args)}
\text{(cons sym (map boolean->cnf args))]}\]
Putting it all together

> (boolean->cnf '(or (and a b) (and (not c) d) (and (not e) f)))
'(and (or (not c) a (not e))
  (or (not c) b (not e))
  (or d a (not e))
  (or d b (not e))
  (or (not c) a f)
  (or (not c) b f)
  (or d a f)
  (or d b f))
SAT Solving

The **satisfiability problem**¹ in computer science asks:

*Given a boolean expression, is there any set of assignments to the variables which results in the equation evaluating to true?*

For example:

- *(and a (not a))*: not satisfiable
- *(and a a)*: satisfiable

(you could imagine much larger inputs)

¹ If you’ve taken algorithms, you probably know that this problem is **NP-complete**
procedure DPLL(e):
    if e is true:
        return true
    if e is false:
        return false
    $v \leftarrow \text{select-variable}(e)$
    $e_1 \leftarrow \text{simplify}(\text{assume-true}(v, e))$
    if DPLL($e_1$):
        return true
    $e_2 \leftarrow \text{simplify}(\text{assume-false}(v, e))$
    return DPLL($e_2$)

Note
DPLL will work with any variable selection from select-variable, but certain
selections may lead to more efficient solutions across the algorithm.
DPLL: Example

\[ a \land (\neg a \lor \neg b) \land c \land (b \lor \neg c) \]

1. Assume \( a \) is true and simplify

\[ \neg b \land c \land (b \lor \neg c) \]

2. \( b = \text{true} \)

false

3. \( b = \text{false} \)

\( c \land \neg c \)

4. \( c = \text{true} \)

false

5. \( c = \text{false} \)

false

6. \( a = \text{false} \)

false

We never reached true, so this equation is not satisfiable

CSCI-400
DPLL: Exercise

Draw the DPLL tree for the following expression, and determine whether the equation is satisfiable or not:

\[(a \lor \neg b) \land (\neg a \lor b) \land (\neg a \lor \neg b)\]
(define (solve-cnf expr)
  (define (solve-rec expr bindings)
    (case expr
      [(#t) bindings]
      [(#f) #f]
      [else
       (let ([sym (choose-symbol expr)])
         (define (solve-assume value)
           (solve-rec (assume sym value expr)
                       (cons (cons sym value) bindings)))
         (let ([sym-true (solve-assume #t)])
           (if sym-true
               sym-true
               (solve-assume #f)))))]))
  (solve-rec expr '()))
choose-symbol

Not a good heuristic, but it works!

(define (choose-symbol expr)
  (if (symbol? expr)
      expr
      (choose-symbol (cadr expr)))))
(define (assume var value expr)
  (cond
   [(eq? var expr) value]
   [(equal? `(not ,var) expr) (not value)]
   [(symbol? expr) expr]
   [else
    (match expr
     [(`not ,_) expr]
      [(list-rest sym args) ...]]))
  ) ;; handle conjunction/disjunction
Handling Conjunction/Disjunction

(let ([look-for (case sym
    [(and) #f]
    [(or) #t]]))
(define (f item acc)
  (if (eq? acc look-for)
      acc
      (let ([result (assume var value item)])
        (cond
          [(eq? result look-for) result]
          [(eq? result (not look-for)) acc]
          [else (cons result acc)])))))
(let ([result (foldl f '() args)])
  (cond
    [(null? result) (not look-for)]
    [(eq? result look-for) result]
Putting It All Together

(define (solve expr)
  (solve-cnf (boolean->cnf expr)))

> (solve '(and a b))
'(b . #t) (a . #t)

> (solve '(or (and a b) (and c d) (and e f)))
'((d . #t) (f . #t) (c . #t))

> (solve '(and a (not a)))
#f

> (solve '(and (or (not a) b) (or a (not b))))
'(b . #t) (a . #t)