

Symbolic Computation

Principles of Programming Languages

Colorado School of Mines

<https://lambda.mines.edu>

LGA Discussion

- 1 What questions did you have on the reading? Can your group members answer, or you can ask me.
- 2 Define **symbolic computation** in your own words.
- 3 What structures in Racket would you find useful for symbolic computation?
- 4 Share what other applications you came up with for symbolic computation. Formulate some more with your group.

Symbolic Computation Defined

- Wikipedia considers symbolic computation to be simply *computer algebra*.
- While computer algebra is a form of symbolic computation, there are plenty of other applications.
 - Programming languages
 - Compilers
 - Artificial intelligence
 - ...

Lisp & Symbolic Computation

Lisp dialects have a **homoiconic syntax**: the code is data, and data is code. Lists being the structure of the language syntax, code can be manipulated just like lists.

- The concept of "quoting" is fairly unique to just Lisp.
- It leads to a natural way to manipulate and work on *code* in the language.
- **Key point:** we can manipulate code before it is evaluated!

John McCarthy (1958)

Recursive Functions of Symbolic Expressions and their Computation by Machine

- Today we will explore a practical application of symbolic computation in artificial intelligence.

Boolean Expressions as S-Expressions

To represent boolean expressions in Racket, we need to formalize an s-expression syntax for them:

Conjunction	$a \wedge b \wedge c \dots$	(and a b c ...)
Disjunction	$a \vee b \vee c \dots$	(or a b c ...)
Negation	$\neg a$	(not a)

Practice: convert to s-expression

- 1 $a \wedge (b \vee c \vee d) \wedge d$
- 2 $\neg a \wedge (a \vee \neg b) \wedge \neg(a \vee b)$

Conjunctive Normal Form

Note

Depending on your background, you may already know this. Bear with me while I explain it to everyone else.

A boolean expression is in **conjunctive normal form** (CNF) if and only if all of the below are true:

- It only contains conjunctions, disjunctions, and negations.
- Negations only contain a variable, not a conjunction or disjunction.
- Disjunctions only contain variables and negations.

Example:

$$(a \vee b \vee c) \wedge (\neg a \vee b)$$

Learning Group Activity

Come up with an expression in CNF (not the example), and one

Verifying CNF in Racket

```
(define/match (in-cnf? expr [level 'root])
  [((? symbol?) _) #t]
  [( `(not ,(? symbol?)) _) #t]
  [((list-rest 'or args) (or 'root 'and))
   (andmap (λ (x) (in-cnf? x 'or)) args)]
  [((list-rest 'and args) 'root)
   (andmap (λ (x) (in-cnf? x 'and)) args)]
  [(- -) #f])
```

Conversion to CNF

We can convert any boolean expression composed of just conjunctions, disjunctions, and negations to CNF using the following mathematical properties:

- **Elimination of double-negation:** $\neg\neg a \rightarrow a$
- **DeMorgan's Law (Conjunction):** $\neg(a \wedge b) \rightarrow (\neg a \vee \neg b)$
- **DeMorgan's Law (Disjunction):** $\neg(a \vee b) \rightarrow (\neg a \wedge \neg b)$
- **Distributive Property:** $a \vee (b \wedge c) \rightarrow (a \vee b) \wedge (a \vee c)$

Practice: Convert to CNF

Convert each expression to CNF:

1 $\neg(a \wedge \neg b)$

2 $\neg((a \vee b) \wedge \neg(c \vee d))$

3 $\neg((a \vee b) \wedge (c \vee d))$

Racket: Convert to CNF

Here's the base structure we want our code to follow:

```
(define (boolean->cnf expr)
  (if (in-cnf? expr)
      expr
      (boolean->cnf
       (match expr
         ...)))) ;; cases for the conversions we know
```

Double Negation Pattern Match

```
[ `(not (not ,e)) e ]
```

Simplify and/or of single argument

```
[ `(or ,e) e ]  
[ `(and ,e) e ]
```

DeMorgan's Law

- DeMorgan's Law for Conjunction

```
[ `(not (and ,@(list-rest args)))  
  `(or ,@(map (curry list 'not) args))] ]
```

- DeMorgan's Law for Disjunction

```
[ `(not (or ,@(list-rest args)))  
  `(and ,@(map (curry list 'not) args))] ]
```

Explosion of and/or with nested expression

- and in and simplification

```
[`(and ,@(list-no-order (list-rest 'and inside) outside) .  
  `(and ,@inside ,@outside)]
```

- or in or simplification

```
[`(or ,@(list-no-order (list-rest 'or inside) outside) ...)  
  `(or ,@inside ,@outside)]
```

Distributive Law

```
[`(or ,@(list-no-order (list-rest 'and and-args) args ...))
  `(or ,@(cdr args)
      (and ,@(map
            (λ (x) (list 'or (car args) x))
            and-args)))]
```

Recurse otherwise...

```
[(list-rest sym args)
 (cons sym (map boolean->cnf args))]
```


Putting it all together

```
> (boolean->cnf '(or (and a b) (and (not c) d) (and (not e) f)
'(and (or (not c) a (not e))
      (or (not c) b (not e))
      (or d a (not e))
      (or d b (not e))
      (or (not c) a f)
      (or (not c) b f)
      (or d a f)
      (or d b f)))
```

The **satisfiability problem**¹ in computer science asks:

Given a boolean expression, is there any set of assignments to the variables which results in the equation evaluating to true?

For example:

- (and a (not a)): not satisfiable
- (and a a): satisfiable

(you could imagine much larger inputs)

¹If you've taken algorithms, you probably know that this problem is **NP-complete**

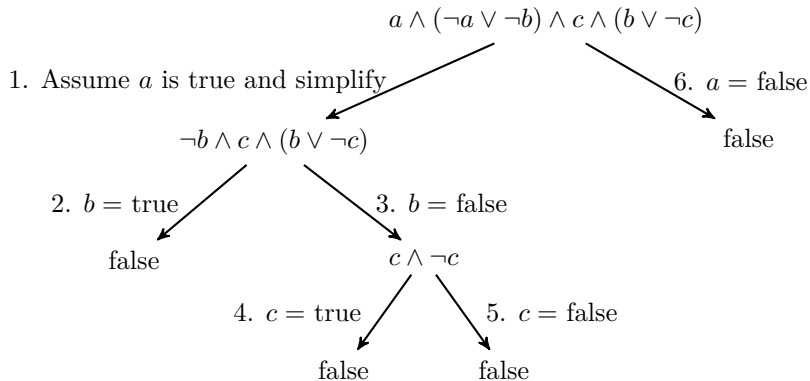
Davis-Puntam-Lodgemann-Loveland Algorithm

```
procedure DPLL( $e$ ):  
  if  $e$  is true:  
    return true  
  if  $e$  is false:  
    return false  
   $v \leftarrow$  select-variable( $e$ )  
   $e_1 \leftarrow$  simplify(assume-true( $v$ ,  $e$ ))  
  if DPLL( $e_1$ ):  
    return true  
   $e_2 \leftarrow$  simplify(assume-false( $v$ ,  $e$ ))  
  return DPLL( $e_2$ )
```

Note

DPLL will work with any variable selection from select-variable, but certain selections may lead to a more efficient solution on average than others.

DPLL: Example



- We never reached true, so this equation is not satisfiable

Draw the DPLL tree for the following expression, and determine whether the equation is satisfiable or not:

$$(a \vee \neg b) \wedge (\neg a \vee b) \wedge (\neg a \vee \neg b)$$

DPLL in Racket

```
(define (solve-cnf expr)
  (define (solve-rec expr bindings)
    (case expr
      [(#t) bindings]
      [(#f) #f]
      [else
       (let ([sym (choose-symbol expr)])
         (define (solve-assume value)
           (solve-rec (assume sym value expr)
                       (cons (cons sym value) bindings)))
         (let ([sym-true (solve-assume #t)])
           (if sym-true
               sym-true
               (solve-assume #f))))))])
  (solve-rec expr '()))
```

choose-symbol

Not a good heuristic, but it works!

```
(define (choose-symbol expr)
  (if (symbol? expr)
      expr
      (choose-symbol (cadr expr))))
```

Assuming and Simplifying

```
(define (assume var value expr)
  (cond
    [(eq? var expr) value]
    [(equal? `(not ,var) expr) (not value)]
    [(symbol? expr) expr]
    [else
      (match expr
        [ `(not ,_) expr]
        [(list-rest sym args)
         ...])])) ;; handle conjunction/disjunction
```


Handling Conjunction/Disjunction

```
(let ([look-for (case sym
                  [(and) #f]
                  [(or) #t])])
  (define (f item acc)
    (if (eq? acc look-for)
        acc
        (let ([result (assume var value item)])
          (cond
           [(eq? result look-for) result]
           [(eq? result (not look-for)) acc]
           [else (cons result acc)]))))))
(let ([result (foldl f '() args)])
  (cond
   [(null? result) (not look-for)]
   [(eq? result look-for) result]
   [else (cons sym result)]))
```

Putting It All Together

```
(define (solve expr)
  (solve-cnf (boolean->cnf expr)))

> (solve '(and a b))
'((b . #t) (a . #t))
> (solve '(or (and a b) (and c d) (and e f)))
'((d . #t) (f . #t) (c . #t))
> (solve '(and a (not a)))
#f
> (solve '(and (or (not a) b) (or a (not b))))
'((b . #t) (a . #t))
```